Serially-Concatenated LDGM Codes for Correlated Sources over Gaussian Broadcast Channels

Mikel Hernaez, Pedro M. Crespo, Javier Del Ser† and Idoia Ochoa

Abstract—We propose the use of serially-concatenated LDGM codes for the transmission of spatially correlated sources over 2-user Gaussian broadcast channels. For this channel it is well-known that the capacity-achieving coding scheme is based on a superposition approach, where the outputs of two independent encoders are modulated with different energies and symbolwise added. This produces a channel sequence that conveys the information from both users to the distant receivers. The use of serially-concatenated LDGM codes with correlated information sources permits to keep a strong degree of correlation in the encoded symbols, which are then coherently added prior to transmission. By properly designing the coding of the correlated sources, the obtained simulation results show that our proposal outperforms the suboptimal fundamental limit assuming separation between source and channel coding.

I. INTRODUCTION

In the Gaussian broadcast channel, first proposed in [1] and further studied in [2], the messages $w_1 \in \{1, 2, \ldots, 2^{nR_1}\}$ and $w_2 \in \{1, 2, \ldots, 2^{nR_2}\}$ generated by two information terminals with rates $R_1$ and $R_2$, are encoded by using a single transmitted signal and sent over two AWGN channels to their corresponding receivers. This channel can be modeled as $Y_1 = X + N_1$ and $Y_2 = X + N_2$, where $X$ denotes its input, $Y_1$ and $Y_2$ the corresponding outputs, and $N_1$ and $N_2$ are arbitrary correlated Gaussian random variables with variances $\sigma_1^2$ and $\sigma_2^2 = \beta \sigma_1^2$. Without loss of generality we will assume $\beta > 1$.

It is well known that the Gaussian broadcast channel belongs to the class of degraded broadcast channels, and its capacity is given by [2]

$$R_1 \leq \frac{1}{2} \log_2 \left( 1 + \frac{\alpha E_c}{\sigma_1^2} \right), \quad R_2 \leq \frac{1}{2} \log_2 \left( 1 + \frac{(1-\alpha) E_c}{\alpha E_c + \sigma_2^2} \right),$$

where $0 \leq \alpha \leq 1$ may be arbitrarily chosen to trade rate $R_1$ for $R_2$ as the transmitter wishes.

To encode the messages, the optimum transmitter generates two codebooks: one with average energy per symbol $\alpha E_c$ at rate $R_1$, and another with energy $(1-\alpha) E_c$ at rate $R_2$. Then, in order to send messages $w_1 \in \{1, 2, \ldots, 2^{nR_1}\}$ and $w_2 \in \{1, 2, \ldots, 2^{nR_2}\}$ to receivers 1 ($Y_1$) and 2 ($Y_2$), the transmitter takes a codeword $X(w_1)$ from the first codebook and a codeword $X(w_2)$ from the second codebook, and computes the sum $X = X(w_1) + X(w_2)$ before sending it over the channel. By obvious reasons, this encoding scheme based on the addition of codewords is usually referred to as superposition scheme.

The receivers must now decode the messages. The receiver associated to the channel $Y_2$ with higher noise variance $\sigma_2^2$ (hereafter bad receiver) looks through the second codebook to find the closest codeword to the receiver vector $Y_2 = X + N_2$. Its effective Signal to Noise Ratio (SNR) is $(1-\alpha) E_c / (\alpha E_c + \sigma_2^2)$, since $X(w_1)$ acts as a noise. On the other hand, the good receiver (i.e. that associated to $Y_1$) first decodes and reconstructs the codeword $X(w_2)$ as $\hat{X}(w_2)$, which can be accomplished due to its lower noise variance $\sigma_1^2$. Then it subtracts this codeword from $Y_1 = X + N_1$, and finally looks for the codeword in the first codebook closest to $Y_1 - \hat{X}(w_2)$.

![Fig. 1. Superposition scheme for the Gaussian broadcast channel.](image)

In this context, several papers (e.g. [3], [4]) have presented practical superposition schemes for independent sources which outperform orthogonal schemes (i.e. time or frequency division multiplexing). However, in dense communication networks (e.g. sensor networks), the physical proximity between nodes leads to a certain degree of correlation between the data registered by the sensors. This correlation should be exploited in reception in order to improve the performance of the communication system. Based in recent results on the correlation-preserving properties of Serially-Concatenated Low-Density Generator Matrix Codes, (SC-LDGM [5], [6]), this letter proposes the use of such codes for the transmission of spatially correlated sources through the Gaussian broadcast channel.

II. PROPOSED SYSTEM

The proposed transmission system is depicted in Figure 2. It is assumed that the spatially correlated multiterminal source $\{U_k, V_k\}_{k=1}^\infty$ is a sequence of independent and identically distributed pairs of binary random variables with distributions $Pr(U_k = 0) = Pr(U_k = 1) = 0.5$ and $Pr(U_k = 0, V_k = 1) = 0.5$. Furthermore, the source symbols $U_k$ and

Manuscript received on June 17th, 2009.

Mikel Hernaez, Pedro M. Crespo and Idoia Ochoa are with the Centro de Estudios e Investigaciones Técnicas de Gipuzkoa (CEIT) and TECNUN (University of Navarra), 20009 San Sebastian, Spain (e-mail addresses: {mhernaez, pcrespo, iochoa}@ceit.es).

Javier Del Ser is with TECNALIA-Robotiker, 48170 Zamudio, Spain (e-mail address: jdelser@robotiker.es).

†: To whom correspondence should be addressed.
\(U_k^2\) are intended for the good and bad receiver, respectively.

Terminal 1 and terminal 2 forms blocks \(U_1 = \{U_1^1\}_{k=1}^{K}\) and \(U_2 = \{U_2^1\}_{k=1}^{K}\) of \(K\) symbols, being processed through two separated identical systematic SC-LDGM codes. Let \(C_1\) and \(C_2\) denote the corresponding output codewords, consisting of the systematic bits \(C_n = U_n\) (for \(n = 1, \ldots, K\)) and the coded (parity) bits \(\{C_n\}_{n=K+1}^{N}\). The reason for using this kind of binary block codes is to preserve, as much as possible, the existing correlation between \(U_1\) and \(U_2\) into the output codewords \(C_1\) and \(C_2\).

More specifically, a rate \(K/N\) systematic LDGM code is a linear binary code with generator matrix \(G = [I \, P]\), where \(I\) denotes the identity matrix of order \(K\), and \(P\) is a \(K \times (N-K)\) sparse matrix. A systematic SC-LDGM code is then built by concatenating two LDGM codes of rates \(K/N_1\) and \(N_1/N\), where the outer code has a rate close to 1 (i.e. \(N_1 \approx K\)). As in [5], [6], we will denote as \((\theta, \vartheta)\) LDGM codes to those codes in which all the \(K\) systematic bit nodes have degree \(\theta\), and each of the \(N-K\) coded nodes has degree \(\vartheta\). In other words, the parity matrix \(P\) of an \((\theta, \vartheta)\) LDGM code has exactly \(\theta\) non-zero entries per row and \(\vartheta\) non-zero entries per column. In the proposed system, the same generator matrices have been used for both encoders. As already mentioned, the advantage of using this type of code is to keep the spatial correlation between the codewords associated to the messages sent by the spatially correlated multiterminal source. This fact is obvious for the systematic part of the codewords. Regarding the parity part of the codewords, it can be shown [7] that the probability for two parity bits, located at the same position in the corresponding two codewords, of being different is

\[
p_c = \frac{1 - (1 - 2p)\vartheta}{2},
\]

where \(p\) and \(\vartheta\) have been previously defined. For very small values of \(p\), (2) can be approximated as \(p_c \approx \vartheta p\). Therefore, by choosing a small \(\vartheta\), the spatial correlation in the parity part of the codewords \(C_1\) and \(C_2\) can be preserved.

![Proposed transmitter for the Gaussian broadcast channel.](image)

Before being added to form the transmitted symbol \(X_k\), the encoded binary symbols \(C_1^k\) and \(C_2^k\) at the output of the two encoders are BPSK modulated to yield \(X_1^k = \sqrt{\alpha E_c} (2c_1^k - 1)\) and \(X_2^k = \sqrt{(1-\alpha)E_c} (2c_2^k - 1)\). Notice that, due to the high level of correlation between \(C_1^k\) and \(C_2^k\), the modulated symbols \(X_1^k\) and \(X_2^k\) will be added coherently with high probability\(^1\). This in turn will improve the detection of the codewords at the corresponding receivers. The downside of this coherent addition is that the actual energy per symbol \(E_{corr}\) sent over the broadcast channel increases, since \(X_1^k\) and \(X_2^k\) are no longer independent. It can be easily shown that \(E_{corr} = E_c(1 + \Delta_{exc})\), where

\[
\Delta_{exc} = R_c \Delta_{sys} + (1 - R_c)\Delta_{par},
\]

\[
\Delta_{sys} = (1 - 2p)2\sqrt{\alpha(1-\alpha)},
\]

\[
\Delta_{par} = (1 - 2p_c)2\sqrt{\alpha(1-\alpha)},
\]

with \(\Delta_{sys}\) and \(\Delta_{par}\) representing the excess energy fraction of the systematic and the parity symbols (parameter \(p_c\) associated to the inner LDGM decoder). In the above expression, the parity symbols introduced by the outer code have been neglected for two reasons: (i) as the outer rate is close to one, the outer parity symbols represent a small fraction of the sent codeword, and (ii) the number of ones per row in the outer parity-check matrix \(H\) is considerably high (e.g. \(\vartheta=76\) in our simulations), thus the correlation is not preserved in these symbols.

### III. Decoding Process

The decoding procedure to estimate \(U_1^k\) and \(U_2^k\) at the corresponding receivers is based on the Sum-Product Algorithm (SPA) applied to the factor graphs that models the SC-LDGM codes [8]. These factor graphs are modified to take into account the existing correlation between terminals.

We begin by analyzing the decoding process of the bad receiver. An estimation of \(U_2\), denoted as \(\hat{U}_2\), is obtained by applying the SPA algorithm over the graph of the SC-LDGM code (\(I\) iterations over the inner code followed by \(I\) iterations over the outer code). The a priori probabilities of the symbols \(U_2^k\) are kept unchanged at 0.5, while the conditional channel probabilities \(p(y_k^2|c_k^2)\) required by the factor graph are given by

\[
p(y_k^2|c_k^2) = \begin{cases} 
(1 - p_{cor})f_{00} + p_{cor}f_{10} & \text{if } c_k^2 = 0, \\
(1 - p_{cor})f_{11} + p_{cor}f_{01} & \text{if } c_k^2 = 1, 
\end{cases}
\]

where \(p_{cor} = p\) for the channel symbols associated to the systematic part of the codeword, and \(p_{cor} = p_c\) for those associated to the parity symbols of the codeword. The \(f_{ij}\) functions are defined as

\[
f_{ij} \triangleq N\left((2i - 1)\sqrt{(1 - \alpha)E_c} + (2j - 1)\sqrt{\alpha E_c}, \sigma_2^2\right),
\]

where \(N(\rho, \sigma^2)\) denotes a Gaussian distribution with \(\rho\) mean and variance \(\sigma^2\). Hence, the correlation between sources is exploited in this receiver not as side information (the a priori probabilities of \(U_2^k\) are unmodified), but intrinsically in the channel conditional probabilities used by the decoder.

As explained in Section I, the good receiver first decodes the bad receiver’s codeword \(C_2\) (\(\sigma_1^2 \leq \sigma_2^2\)) and then, after an appropriate scaling, subtract it from the received signal, i.e.

\[Y_1 - \sqrt{(1 - \alpha)E_c}C_2,\]

Based on this sequence, and having an effective signal to noise ratio \(\alpha E_c/\sigma_2^2\), the receiver obtains an estimation \(\hat{U}_1\) for \(U_1\). In this case the a posteriori probabilities of the symbols \(U_1^k\) are modified by the a posteriori probabilities of the symbols \(U_2^k\), introduced here as side information, i.e.

\[p(u_1^k) = p(u_1^k | u_2^k = 0)p(u_2^k = 0) + p(u_1^k | u_2^k = 1)p(u_2^k = 1),\]

On the other hand, the conditional channel probabilities are now given by \(N(\sqrt{\alpha E_c}, \sigma_1^2)\).
IV. SIMULATION RESULTS

In order to assess the performance of the proposed system, Monte Carlo simulations have been done for different values of \( p \) and scale factor \( \alpha \). A serially concatenated LDGM code has been used for both terminals, composed by a (4,76) regular outer LDGM code and a (14,7) regular inner LDGM code. The resulting overall code rate is \( R_e \approx 0.316 \). The block size for all the simulations is kept fixed to \( K = 9500 \), and \( T = 100 \) iterations have been considered for the SPA decoding algorithm. The ratio between \( \sigma_1^2 \) and \( \sigma_2^2 \) is set to \( \beta = 3 \).

\[
\frac{E_c}{\sigma_1^2} = \frac{2^R_e H(U^1|U^2) - 1}{\alpha_\star},
\]

with \( \alpha_\star \) given by

\[
\alpha_\star = \frac{2^{2R_e} H(U^1|U^2) - 1}{\beta (2^{2R_e} H(U^2) - 1) - 2^{2R_e} H(U^2) + 2^{2R_e} H(U^1|U^2)},
\]

where, from our multiterminal source assumption, we have that \( H(U^2) = 1 \) and \( H(U^1|U^2) = H(p) \), i.e. the entropy of a binary random variable with distribution \( (p, 1-p) \).

Figure 3 plots the Bit Error Rate (BER), computed by averaging the BER at both receivers, versus the gap in dB to the separation-based limit for \( p = 0.1 \) (left) and \( p = 0.01 \) (right). That is,

\[
\text{Gap} = 10 \log_{10} \frac{E_{c,\text{corr}}}{\sigma_1^2} - 10 \log_{10} \left[ \frac{E_c}{\sigma_1^2} \right]_{\min} \text{ (dB)},
\]

where \( E_{c,\text{corr}}/\sigma_1^2 \) denotes the SNR required by our proposed system to achieve the corresponding BER level. Several values for the energy-splitting parameter \( \alpha \) are considered (see Figure 2). Notice that in all simulated curves the proposed system outperforms the separation-based limit (e.g. around \( 1.75 \text{ dB at BER} = 10^{-3} \) for \( p = 0.1 \)). For \( p = 0.1 \) observe that the best waterfall performance is achieved by using a unique value of \( \alpha_\star(0.24) \), behavior that also holds for \( 0.25 \geq p \geq 0.1 \). However, for high correlation levels \( (0.05 \geq p > 0) \) the selection of \( \alpha \) is a tradeoff between the error floor level and the waterfall region (Figure 3, right).

<table>
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<tr>
<th>( p )</th>
<th>0.25</th>
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<th>0.15</th>
<th>0.1</th>
<th>0.10</th>
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<td>-0.98</td>
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TABLE I
GAPS FOR SEVERAL VALUES OF THE CORRELATION PARAMETER.

Finally, Table I shows the \( \text{Gap} \) performance for different values of \( p \). Also shown in the table are the corresponding optimal \( \alpha_\star \)'s. Again, in all cases the separation-based limit is outperformed. It is interesting to observe the gap inflection point at \( p = 0.1 \). This is due to the fact that, for a given \( E_c \), increasing the correlation level (i.e., decreasing \( p \)) will increase the coherence effect and, in turn, the BER performance. However, at the same time the effective \( E_{c,\text{corr}} \) will also increase. There is a point (around \( p = 0.1 \)) where the latter effect outgains the first effect.

V. CONCLUSION

We have proposed a superposition system for the transmission of two correlated sources over a Gaussian broadcast channel. Each information sequence is independently encoded using SC-LDGM codes, which allows preserving the correlation between the sources, and ultimately leads to a coherent signal addition at the superposition stage. The proposed system is able to outperform the fundamental limit assuming separation between source and channel coding.

REFERENCES