A Novel Sequential Bayesian Approach to GPS Acquisition

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Abstract—In this work, a novel online learning algorithm is presented for the synchronization of Global Positioning System (GPS) signal parameters at the acquisition, or coarse synchronization, stage. The algorithm is based on a Bayesian approach, which has, to date, not been exploited for the acquisition problem. Simulated results are presented to illustrate the algorithm performance, in terms of accuracy and acquisition time, along with results from the acquisition of signals from live GPS satellites using both the new algorithm and a state-of-the-art approach for comparison.

I. INTRODUCTION

For useful data to be read from code division multiple access (CDMA) signals, such as those used by Global Navigation Satellite Systems, including GPS, the receiver must, first, synchronize the received signal with a locally generated replica signal, in order that the pseudo-noise (PN) code and carrier frequency components can be removed, leaving only the data component. This synchronization process is carried out in two steps: namely acquisition (coarse synchronization) and tracking (fine synchronization). The parameters to be estimated are the time delay of the PN code, $\hat{\zeta}$, and the Doppler frequency offset, $\hat{\omega}_D$.

The current state-of-the-art for acquisition is to perform a two-dimensional grid search with discrete parameter pairs, $\hat{\vartheta} = \{\hat{\zeta}, \hat{\omega}_D\}$ [1]. For each pair, correlation is carried out between a locally generated replica signal with the candidate parameters and the received signal. Due to the high autocorrelation and low cross-correlation properties of the PN codes (which, for the GPS civil L1 C/A signal, are Gold codes [2]), the magnitude of the correlator output, in the absence of noise, is large when the local replica and true parameters are synchronized and is negligible when the local replica parameters are not matched with the true parameters. Thus, the maximum value of the correlator output across the two-dimensional grid can be chosen as the correct parameter pair, and the magnitude of the correlator output compared with a threshold to decide whether or not synchronization has, in fact, been achieved [3]. When the received carrier-to-noise ratio ($C/N_0$) is low, as is often the case for receivers indoors or in urban canyons, further processing is required to increase the signal power gain so as to distinguish between signal and noise. Alternatively, some very recent work has proposed using probabilities, derived from the correlator outputs in the discrete “cells” (parameter pairs, $\hat{\vartheta}$), as decision metrics [4], [5].

In order for a reliable position estimate to be calculated by the GPS receiver, signals from a minimum of four satellites must be acquired [6]. Thus, efficient acquisition performance is vital to receiver operation. In the GPS literature, much effort has been focused on reducing the acquisition time, most significantly by processing multiple estimates simultaneously (parallel search) instead of considering each candidate separately (serial search). In hardware implementations, banks of correlators can be employed, each matched to a different part of the PN code. In the case of software implementations, the most common approach is to simultaneously calculate the correlation value at all code delays using a Fast Fourier Transform (FFT) [7]. Hybrid parallel/serial systems have also been developed to provide a trade off between computational load and acquisition time performance [8].

In this work, we present a novel method for the acquisition of GPS signals, approaching the problem from a Bayesian perspective to exploit the known shapes of the probability distribution function of the correlator in the presence and absence of synchronization. In a nutshell, if the received and local signals are synchronized, the distribution is a non-central chi-squared distribution with a known mean, which is a function of the residual Doppler frequency. A grid search ignores this information and we intend to exploit it with a Bayesian model to be able to provide a more accurate prediction for the Doppler shift and synchronize the signals using less correlated samples. The algorithm which is presented is an sequential algorithm, which determines the correct signal parameters based on all of the information collected to date.

The paper is laid out as follows: in Section II, the GPS acquisition problem is outlined, with emphasis on the signal model and the current state-of-the-art, followed by an introduction to the problem in a Bayesian framework, highlighting the background theory for the new approach. The novel algorithm is described in Section III along with some illustrative examples showing its operation. Simulated results and performance characteristics from real GPS signals are presented in Section IV, showing that the new algorithm has a better performance, in terms of mean acquisition time and accuracy of the Doppler estimate, than traditional methods. Finally, conclusions are outlined in Section V.
II. GPS ACQUISITION

A. Signal structure and model

GPS signals are CDMA signals consisting of three primary components, namely the PN code, a radio frequency (RF) carrier and a data signal, which holds the useful information including, for example, the satellite position. Each GPS satellite, of which there are 32 currently operational [6], uses a different PN code, chosen from the family of Gold codes. At any given time, there are typically six to eight satellites in the line-of-sight (LOS) of a receiver which has a clear view of the sky.

The signal model typically used for studying CDMA acquisition is a complex baseband model, representing the received signal, $r(t)$, after down-conversion of the carrier frequency to baseband [9]:

$$r(t) = s(t) + n(t) = \sum_{k \in S_{sv}} A_k d_k(t)c_k(t - \tau_k) \exp\left(j(\omega_{D,k}t + \phi_k)\right) + n(t),$$

where $S_{sv}$ is the set of all received satellites in view and, for the $k$th signal, $A_k$ is the signal amplitude, $c_k$ is the spreading code, which is unique for each satellite, $d_k$ is the data signal, $\omega_{D,k}$ is the Doppler uncertainty frequency (rad s$^{-1}$), $\tau_k = \xi_k T_{\text{code}} + \zeta_k T_{\text{chip}}$ is the time delay consisting of an integer number, $\xi_k$, of full code periods, (for GPS, $T_{\text{code}} = 1$ ms), and $\zeta_k$ chip periods, (for GPS, $T_{\text{chip}} = \frac{1}{1023}$ ms), $\phi_k$ is the initial carrier phase offset (rad) and $n(t)$ is an AWGN noise term with two-sided power spectral density, $N_0/2$ (W/Hz). Although there exist algorithms which process multiple signals simultaneously to speed up the acquisition process, e.g. [10], in this work, as in most of the work in the field, we consider each received signal separately in the acquisition process so the subscript, $k$, is henceforth neglected. This method allows the acquisition process to, first, focus on the satellite with strongest signal power and, once it has been acquired, it can be cancelled from the received signal and subsequent satellites can be acquired in order of decreasing signal power, thus helping to alleviate the problem of intra-system interference, e.g. [6], [1].

B. State-of-the-art

The state-of-the-art in CDMA acquisition, is to correlate the received signal samples with a locally-generated replica signal with parameters taken from a two-dimensional grid of discrete Doppler frequency and code phase offset values [1]. As has been previously mentioned, multiple parameter pairs, or “cells”, may be tested simultaneously by parallelization in the time domain or the frequency domain.

The correlator output, $y$, as a function of the estimated and true signal parameters, $\hat{\theta}$ and $\theta$, respectively, is given by:

$$y(\hat{\theta}) = \left| \sum_{m=0}^{M-1} r(mT_s) \exp(-j\hat{\omega}_D m T_s)c(mT_s - \hat{\zeta} T_{\text{chip}}) \right|^2,$$

$$= M N_s A^2 \left| R(\delta\zeta) a_D(\delta\omega_D) + \text{noise} \right|^2,$$

where $M$ is the number of code periods coherently summed in the correlator, $N_s$ is the number of samples per PN code period with sample time, $T_s$, $R(\delta\zeta)$ is the auto-correlation function of the PN code with offset $\delta\zeta = \zeta - \zeta$, modeled by:

$$R(\delta\zeta) = \begin{cases} |1 - \delta\zeta|, & \text{if } |\delta\zeta| < 1 \\ \epsilon, & \text{otherwise} \end{cases}$$

the attenuation, $a_D$, as a function of the residual Doppler shift, $\delta\omega_D = \omega_D - \hat{\omega}_D$. (4) The magnitude of the worst case residual, $\Delta\omega_D$, between the two samples closest to the peak is dependent on the resolution of the grid in the Doppler frequency domain with a total main lobe width of $2/M_c$ kHz. The resolution can be improved by increasing the value of $M_c$; however, this causes a linear increase in the number of cells in the grid as well as in the integration time in each cell and, hence, results in an increase of a factor, $M_c^2$, in the mean acquisition time. With this in mind, $M_c$ will be fixed at one code period throughout this work.

Further to increasing the coherent integration period, $M_c$, the receiver gain can also be increased by non-coherent or differentially coherent summations of the coherent outputs. The total dwell length (including both coherent and non-coherent integration) can be a fixed or variable quantity but, in general, variable dwell lengths offer better performance in terms of reduced mean acquisition time for a given probability of correct detection and/or false alarm. In recent work [11], [12] it has been shown that optimization of dwell parameters, such as coherent and non-coherent integration time and decision thresholds, in variable dwell time strategies, can help to bring the mean acquisition time performance close to that of Wald’s sequential probability ratio test (SPRT), which is known to be optimal [13].

The main problem associated with using the optimal SPRT and, indeed, variable dwell time strategies in general, is that, whilst they perform well in a single cell (i.e. for a single parameter pair, $\hat{\theta}$), they are not suitable for the initial search throughout the entire search space, in which an extremely large number of cells must be searched (e.g. 750000 cells [1]). Thus, an efficient initial search stage, using parallel methods and a
threshold-crossing criterion, is usually carried out and a small number of candidate cells, whose correlator outputs exceed the initial threshold, are passed to a verification stage where the variable dwell time strategies are effective [14], [15]. The method proposed in this work aims to extend the variable dwell time approach to the entire search space without the loss of efficiency suffered by traditional methods. Furthermore, our method avoids the situation where the correct cell is missed by the initial search stage, in which case successful acquisition cannot be achieved, even with an excellent verification strategy [5].

In a recent publication, Pany et al. ([1]) presented a comparison of state-of-the-art methods in hardware and software, showing that, with the generalized likelihood ratio detector, which is “optimal” in an empirical sense, signals with extremely weak C/N₀ (as low as 17 dB-Hz) can be successfully acquired. This method is employed in Section IV to compare state-of-the-art results with the results of the new algorithm developed in this work.

C. Bayesian approach

Exploiting all of the knowledge we have about the distribution of the noisy correlator output, to improve the accuracy of acquisition outputs, has not previously appeared in the literature. By posing the problem in a Bayesian framework, the posterior probability distribution of the Doppler frequency, given correlator outputs evaluated at sample points throughout the search space can be evaluated and used to estimate the Doppler frequency. Among other benefits, this approach eliminates the need to sum multiple correlator outputs (coherently, non-coherently, or otherwise) within a single cell, which is inherent in the traditional algorithms.

We can compute the posterior for the Doppler frequency, \( \omega_D \), as:

\[
f(\omega_D, \zeta_{\text{max}}|y, \omega, \zeta) = \frac{f(y|\omega_D, \omega, \zeta_{\text{max}}, \zeta)f(\omega_D)f(\zeta_{\text{max}})}{f(y|\omega, \zeta)},
\]

where \( \omega = \{\omega_i\}_{i=1, \ldots, n} \) is the set of \( n \) sample Doppler values, \( \zeta = \{\zeta_i\}_{i=1, \ldots, n} \) is the set of \( n \) sample code phase values, \( y = \{y_i\}_{i=1, \ldots, n} \) is the set of corresponding correlator outputs, which have been normalized to give unit noise variance, \( \zeta_{\text{max}} \) is the true value of the code phase offset and \( \omega_D \) is the value of the true Doppler frequency shift. The priors, \( f(\omega_D) \) and \( f(\zeta_{\text{max}}) \), can take various forms, depending on the availability of prior information regarding the signal parameters; in the case of “cold” acquisition (no prior information), a uniform prior should be assumed for both variables. Both priors are independent because there is no relationship between the Doppler shift and the code phase delay.

For a coherent integration over \( M_c \) code periods and non-coherent summation of \( K \) coherent outputs, the distribution of each correlator output is either a central or non-central chi-squared distribution. If the received and local signals are not synchronized in both code phase and Doppler dimensions, it is a central distribution; otherwise it is non-central with a non-centrality parameter given by \( \lambda_i \):

\[
\lambda_i = 0.25K(M_cN_0A)^2R(\delta \xi_i)^2a_D(\delta \omega_D,i).
\]

Fig. 1 shows the dependence of \( \lambda \) on \( \delta \xi \) and \( \delta \omega_D \). Thus, the likelihood for each independent sample is given by:

\[
f(y_i|\omega_D, \omega_i, \xi, \zeta, \zeta_{\text{max}}) = \left(\frac{y_i}{\lambda_i}\right)^{0.5(K-1)} \exp\left(-y_i - \lambda_i\right) \times I_{K-1}(2\sqrt{y_i\lambda_i}),
\]

where \( I_{\nu}(\cdot) \) is the \( \nu \)th order modified Bessel function of the first kind [16]. Whilst (7) gives the general form of the pdf of the correlator output, throughout the rest of the work, \( K = 1 \) is assumed, hence simplifying the pdf to:

\[
f(y_i|\omega_D, \omega_i, \xi, \zeta, \zeta_{\text{max}}) = \exp\left(-y_i - \lambda_i\right) I_0(2\sqrt{y_i\lambda_i}).
\]

III. SEQUENTIAL ALGORITHM

The algorithm presented in this work is an online learning algorithm, which jointly uses all of the samples evaluated to date to estimate the signal parameters, instead of discarding samples once a non-synchronized decision has been made about one particular cell. Thus, the algorithm incrementally improves its knowledge of the entire search space and its frequency resolution, as it is not limited to pre-defined sampling points. The following commonly used assumptions have been made for the purposes of algorithm design:

1) The received carrier-to-noise power ratio is known to the receiver.

2) The PN code of the satellite in view is known to the receiver.

Assumption 1 is made for the purposes of evaluating the probabilities at each step; if the \( C/N_0 \) is, in fact, lower than
the assumed value, a performance degradation will occur. Assumption 2 is based on the fact that the positions of satellites in space at any given instant is predefined and, thus, which of the satellites are in view is, typically, known a priori. Throughout the algorithm, an FFT (or similar) can be used to simultaneously evaluate the correlator output in all code delays, although a method for the parallel evaluation of the posterior distribution of the parameters is not currently available.

A. Basic Algorithm

The main steps of the basic algorithm are outlined below:

**Initialization**: Generate a set of \( n_{\text{init}} \) samples: \( \mathbf{x} = \{\omega_1, y_1\}_{i=1}^{n_{\text{init}}} \), where \( y_i \) are the correlator outputs evaluated at values of the Doppler parameter, \( \omega_i \), chosen from a uniform grid throughout the search space.

Set \( n = n_{\text{init}} \).

**Iterations**: 

1) Calculate posterior probability for each code delay estimate, \( \zeta \), and find the argument of the maximum value, \( \zeta_{\text{max}} \).

2) Calculate the expected value of \( \hat{\omega}_D \): 
   \[
   \hat{\omega}_D = \int_{-\Omega/2}^{\Omega/2} \omega_D f(\omega_D|y, \omega, \zeta_{\text{max}}) d\omega_D.
   \]

3) Evaluate the variance of the estimate: 
   \[
   \text{Var}[\hat{\omega}_D] = \int_{-\Omega/2}^{\Omega/2} \omega_D^2 f(\omega_D|y, \omega, \zeta_{\text{max}}) d\omega_D - \hat{\omega}_D^2.
   \]

4) Decision:
   - If \( \sqrt{\text{Var}[\hat{\omega}_D]} \leq V_{th} \), Terminate the algorithm with \( \omega_D = \hat{\omega}_D \).
   - If \( \sqrt{\text{Var}[\hat{\omega}_D]} > V_{th} \), Generate a new sample, \( \omega \), from the search grid and calculate the equivalent correlator output, \( y_{n+1} \), set \( n = n + 1 \) and return to step (1).

The value of the decision “threshold”, \( V_{th} \), can be set at the desired standard deviation in the estimate at the termination of the algorithm. As such, the threshold value reflects the desired resolution of the estimate in the Doppler dimension. The presence of a heuristic threshold in the algorithm is simply a means to compare the performance to current state-of-the-art algorithms.

B. Performance

A brief description of how the algorithm performs is provided, in particular, in terms of the effect of different \( C/N_0 \) levels and varying threshold value, \( V_{th} \). Fig. 2a shows an example of the estimated \( \hat{\omega}_D \) vs. the number of samples for a high \( C/N_0 \) (40 dB-Hz). It can be seen that the estimated Doppler frequency rapidly converges to the correct value with the variance of the estimate also decreasing rapidly when the estimate approaches the correct value. After this point, the rate of decrease of the standard deviation slows so that improved accuracy requires a significant increase in acquisition time.

In the example shown, the code phase delay was unknown to the receiver so the results refer to a full two-dimensional search throughout a search space of \( \omega_D \in [-2\pi(5780) \text{ Hz}] \) and \( \zeta \in \{0, T_{\text{code}}\} \). Fig. 2b shows a similar example for \( C/N_0 = 25 \text{ dB-Hz} \). Clearly, the lower \( C/N_0 \) requires an increased number of samples but it, also, is seen to successfully converge on the correct \( \omega_D \) value.

The posterior distribution for \( \omega_D \) at the termination of the algorithm for a similar example with \( C/N_0 = 30 \text{ dB-Hz} \) is shown in Fig. 3. The standard deviation of the posterior estimate reached the threshold of 200 Hz at the 190th sample in this example. The posterior distribution with the correct code phase delay is seen to have a single distinct peak, which coincides with the true Doppler frequency (\( \omega_D = 21802.2 \text{ rad/s} \)). For the incorrect code phase delay, on the other hand, there are multiple peaks with low posterior probabilities and the variance of the posterior estimate is much larger over the
Doppler range.

IV. ACQUISITION RESULTS

In order to compare the algorithm with the current state-of-the-art, the following are the primary performance characteristics to be considered:

- Total acquisition time: $T_{ACQ} = nMCT_{code}$.
- Root mean square error of Doppler estimate (dependent on the stopping threshold, $V_{th}$).
- Probability of incorrect code delay estimate.

A. Simulation

Acquisition of simulated GPS signals at varying signal-to-noise ratios and with varying signal parameters ($\zeta$ and $\omega_D$) was carried out to analyze the performance of the new algorithm. The simulated signals differ from real GPS data in that they do not contain a data signal component and there is only one satellite in view in each signal. However, the performance in the presence of the data component and interfering signals is shown to be similar in Section IV-B, validating these simplifications. It should be noted that in all of the simulated trials, the estimated code delay was found to be correct.

Fig. 4a shows the root mean square error in the Doppler estimate, $RMSE = \sqrt{E[(\omega_D - \hat{\omega}_D)^2]}$, vs. the stopping threshold, $V_{th}$, with various $C/N_0$ values ($E[x]$ denotes the expected value of the variable, $x$). It is clear from the graph that increasing the required certainty in the estimate (i.e. decreasing $V_{th}$) gives improved resolution in the Doppler domain, with the RMSE seen to be less than or equal to the standard deviation of the posterior distribution of the estimate in each case. The improvement is particularly significant for thresholds of 100 Hz or less. A resolution of less than 100 Hz, using conventional methods, would require a coherent integration time of at least 10 ms, introducing severe losses due to integration over a data bit boundary on average once for every four integration periods (assuming a bit change probability of 50%).

The effect of the $C/N_0$ and varying $V_{th}$ on the mean acquisition time, $T_{ACQ} = E[n]MCT_{code}$, is shown in Fig. 4b. It can be seen that the afore-mentioned improvement in Doppler resolution comes at the expense of an increase in the acquisition time. Furthermore, as would be expected, lower values of $C/N_0$ result in longer acquisition times. The increase in acquisition time for thresholds between 500 and 100 Hz can be seen, in Fig. 4a, to yield less significant resolution improvements, than that for thresholds less than 100 Hz. Thus, it is argued that a threshold of 100 Hz or less should be chosen in practice, as a trade off between resolution and acquisition time.
B. Real GPS data

Acquisition tests on GPS signal data collected at the University of Calgary, Canada, were carried out to validate the performance of the simulated results and to compare the algorithm with state-of-the-art results. The data were collected in a rooftop environment which is known to have low multipath and a clear view of the sky in all directions. Low C/N₀ conditions were simulated in a controlled manner by use of a signal attenuator on the receiver. A full description of the data collection can be found in [17].

The real signals were acquired using the state-of-the-art method of Pany et al. ([1]), whose parameters were designed to yield a false alarm probability of 1% per satellite. The parameters were designed for the detection of extremely weak signals (an average sensitivity of 17 dB-Hz) which are not present in the collected data. Hence, a fair comparison between the acquisition times of the new algorithm and this state-of-the-art method cannot directly be made for stronger signals (25 - 30 dB-Hz), as the integration times in [1] (K = 200, Mₚ = 16) are excessive for such C/N₀ levels. However, it will be demonstrated that an improved Doppler resolution is attained by the new algorithm.

The method of [1] assumes a Doppler resolution of 0.0319 Hz; however, this is an extreme case due to the very high processing gain levels. However, it illustrates that even with a bin width of 62.5 Hz has been used, the residual offset at termination of the algorithm can be up to two bin widths from the true value. Passing such an estimate to the tracking loops will hinder the fine synchronization performance, as the correlator output will be highly attenuated relative to its peak value.

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