Weighted Sum Rate Maximization for the MIMO X Channel through MMSE Precoding

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Abstract—Recent results elucidate the optimality of interference alignment concept for attaining the degrees of freedom of the MIMO X channel. This criterion is useful in the high SNR regime, but in the low-medium SNR regime, optimizing the weighted sum-rate is more meaningful. Moreover, MIMO X channel subsumes the MIMO interference channel, MIMO multiple access, dual MIMO point-to-point and MIMO broadcast channel. In this respect it is desirable to have an algorithm to design the proper linear transmitters and receivers in all cases. In this work we have observed that an algorithm based on alternate optimization along with the proper initialization is able to provide MMSE precoders attaining significant gains in terms of SNR offset with respect the conventional interference alignment solution.

Keywords: degrees of freedom, interference alignment, MMSE precoders, MIMO X channel

I. INTRODUCTION

With the increase the worldwide data traffic demand due to the penetration of smartphones and internet-based social networks, wireless cellular network designers are requested to find more efficient transmission techniques. A fruitful path of improvement is the inclusion of multi-antenna terminals (MIMO), transforming the conventional scenarios to the point-to-point MIMO, MIMO multiple-access channel (MAC), MIMO broadcast channel (BC) and MIMO interference channel (IC). The benefits of those channels can be characterized by the degrees of freedom (DoF), which measure how the system sum-rate scales in the high power regime. In all cases the DoF can be attained using zero-forcing (ZF) or Minimum Mean Square Error (MMSE) techniques. However, this solution does not allow exchanging the available DoF between the different messages as it is illustrated in the following example.

Motivation example: Let us assume that certain MIMO X channel configuration has 4 DoFs, so that messages can be transmitted and each MS can receive up to 2 messages. The optimum transmission scheme provides a single precoder for each message $m_j$ in Fig. 1. However, how should these precoders be modified depending on channel gains or message priorities? For instance, consider the limit cases:

1. Cross-channels gains are null (dual MIMO point-to-point), or
2. There is only one active source (MIMO-BC)
3. There is only one active destination (MIMO-MAC)
4. The priority of messages $m_{11}$, $m_{22}$ (or $m_{12}$, $m_{21}$) are set to zero (MIMO-IC)

We focus to the MIMO X channel as presented in Fig. 1, with two sources and two destinations. The interest of characterizing the X channel lies in the following fact: MIMO point-to-point, MIMO-IC, MIMO-MAC and MIMO-BC are particular cases the MIMO-X channel. For instance, having 3 antennas at all terminals the DoF of the former is three, [1][3], while MIMO X channel attains four [4]. When all terminals have the same number of antenna, the DoF are analyzed in [4] and [5]. The general MIMO case is solved in [6], showing that outer bounds are achievable with a precoder design based on the Generalized Singular Value Decomposition (GSVD) [7][8]. IA and transmit ZF play a key role in deriving the optimum precoding scheme.

Nevertheless, the IA schemes given in [4][5][6] are designed to maximize the DoF assuming equal-priority messages to both destinations. One option to combat such drawback is: once derived the right precoders that align the interference, then let us apply an additional transmit covariance matrix to maximize the respective objective function. However, this solution does not allow exchanging the available DoF between the different messages as it is illustrated in the following example.

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It is clear that in case 1, we should allow to share the same signal space by the desired signal and the weak interference, e.g. imposing a MMSE receiver. Unfortunately, that is not possible by definition when IA is considered. Additionally, at low and medium SNR there is no point in studying the DoF, and a more meaningful measure of performance is the weighted sum rate.

MMSE precoders and receivers have been proposed in [10] for the MIMO BC. Quality of service (QoS) is tackled by deriving the proper transmit precoders that maximize the weighted sum-rate (WSR) assuming MMSE receivers. The problem is non-convex and only local optimum points can be guaranteed. A reasonably good solution is obtained by alternate optimization of the transmit filters given certain receive filters and afterwards, optimizing the receive filters for the given transmitters [9]. The initialization of this algorithm influences the final outcome of the problem due to its non-convexity.

The MIMO X channel presents some differences with respect to the MIMO BC channel which imply an adequate adaptation of the work investigated in [10]. Still, the problem obtained is non-convex, but a meaningful initialization is possible by adopting the solution provided in [6] based on IA and transmit ZF. Although we elaborate on the results of [10][6], the resulting algorithm for designing transceivers and receivers feature significant desirable properties:

- Transmit precoders and receive filters adapt to the current channel configuration and user priorities in order to maximize the WSR, and become the IA solution at high SNR.
- The maximum DoF per transmitted message are provided by the proposed algorithm, not only for the X channel but also for the MIMO-BC, dual MIMO point-to-point, MIMO-MAC and MIMO-IC.
- In the low-medium SNR regime it is observed a significant enhancement in terms of SNR offset of the proposed solution over the IA zero forcing solutions.

II. SYSTEM MODEL

The MIMO X channel depicted in Fig.1 consists of two sources and two destinations, where each source is transmitting independent messages to all destinations. Hence, each destination is receiving multiple messages from both sources. Each source is equipped with $M$ transmitting antennas, while destinations have $N$ antenna elements $(i,j \in \{1,2\})$. Let us denote the message transmitted from the $i$th source to the $j$th destination by $m_{ij}$ consisting of the symbol vector $d_{ij}$. That message employs the transmit filter matrix $B_{ij}$. The signal model at the $j$th destination is given by,

$$y_{j} = [H_{ij} H_{ji}] \begin{bmatrix} B_{i1} & B_{i2} \\ 0 & 0 \end{bmatrix} d_{ij} + n_{j}$$

(1)

where $y_{j}$ and $n_{j}$ are the received signal and additive white Gaussian noise seen by the $j$th destination, respectively, $H_{ij} \in \mathbb{C}^{N \times M}$ stands for the channel matrix between the $i$th destination and the $j$th source, finally $d_{ij} = [d_{ij}^T d_{ij}^T]^T$.

The effective noise covariance matrix when symbols $d_{ij}$ have to be decoded is given by,

$$R_{w(j)} = H_{j} \left( \sum_{k=1,k \neq j}^{2} B_{kj} B_{kj}^H \right) H_{j}^H + \sum_{n=1,n \neq j}^{2} H_{n} \left( \sum_{k=1}^{2} B_{kn} B_{kn}^H \right) H_{n}^H + \sigma^2 I_N$$

(2)

where the $\sigma^2$ accounts for the AWGN noise power at the $j$th destination, the second term in (2) denotes the interference due to messages transmitted by the $j$th source but intended to other user $(k \neq j)$ and the last term stands for the generated interference due to messages transmitted by a different source, with $(n \neq j)$.

Under the assumption of independent message decoding, the achievable rate for message $m_{ij}$ is given by,

$$R_{ij} = \log_{2} \det\left( I + B_{ij}^H R_{w(j)}^{-1} B_{ij} \right)$$

(3)

If linear receive filters are envisioned the symbols are decoded as,

$$\hat{d}_{ij} = A_{ij}^H y_{j}$$

(4)

with $A_{ij}$ the linear receiver filter for the $m_{ij}$-th message and $y_{j}$ the received signal introduced in (1). In case of a MMSE receiver, the linear filter becomes,

$$A_{ij} = B_{ij}^H \left( H_{ij} B_{ij} B_{ij}^H H_{ij}^H + R_{w(j)} \right)^{-1}$$

(5)

with $R_{w(j)}$ defined in (2). In this regard, the MSE-matrix for message $m_{ij}$ due to the MMSE receive filter is [11] connected with the achievable rate

$$E_{ij} = E\left[ (d_{ij} - \hat{d}_{ij}) (d_{ij} - \hat{d}_{ij})^H \right] = \left( I + B_{ij}^H R_{w(j)}^{-1} B_{ij} \right)^{-1}$$

$$R_{ij} = \log \det\left( E_{ij}^{-1} \right)$$

(6)

III. MAIN OBJECTIVE

This work pursues the maximization of the weighted sum-rate (WSR) of the system depicted in Fig.1 using the transmitter filters that solve this optimization problem,

$$\left( P_{\text{WSR}} \right) \min_{\mu_{ij}} \sum_{i,j} \mu_{ij} R_{ij}$$

s.t. $\text{tr}\left( B_{ij}^H B_{ij} + B_{ji}^H B_{ji} \right) = P_{j}$ $j \in \{1,2\}$

(7)

where $\mu_{ij}$ denotes the priorities given to message $m_{ij}$ while $R_{ij}$ accounts for bitrate defined in (3). Finally, $\text{tr}$ stands for the trace operator. $P_{\text{WSR}}$ problem is hard to be handled due to its non-convexity. Nevertheless, following similar steps derived in [10] for the MIMO BC, we are able to show that the KKT conditions of the WSR and new optimization problem based on...
the minimization of the weighted minimum mean square error (WMMSE) have a simple relationship. The WMMSE optimization problem is defined by,

\[
\begin{align*}
\mathbf{P}_{\text{WMMSE}} &= \min_{\mathbf{W}_j} \sum_{i,j} \text{tr}(\mathbf{W}_j \mathbf{E}_{ij}) \\
\text{s.t.} \quad &\text{tr}(\mathbf{B}_j \mathbf{B}_j^H + \mathbf{B}_j \mathbf{B}_j^H) = P_j, \quad j \in \{1, 2\}
\end{align*}
\]

where \(\mathbf{E}_{ij}\), \(\mathbf{W}_j\) denote the MSE-matrix defined in (6) and a given MSE-weight matrix, respectively. It turns out that the (\(\mathbf{P}_{\text{WSR}}\)) WSR-gradient and the (\(\mathbf{P}_{\text{WMMSE}}\)) WMMSE-gradient are identical for a given transmit filters \([12]\) and modifying the MSE criterion in the MIMO BC can be described in closed-form (PWSR) WSR-gradient and the (P WMMSE) WMMSE-gradient are given MSE-weight matrix, respectively. It turns out that the MMSE matrices \(\mathbf{W}_{ij}\), \(\mathbf{W}_{ij}^H\) have a simple relationship. The WMMSE optimization between the WMMSE, MSE-weights (computed at each iteration) and the proper receiver update to be described in section V.

IV. WEIGHTED MMSE OPTIMIZATION

The transmit precoders that minimize the WMMSE criterion in the MIMO BC can be described in closed-form assuming a given receive filters \([12]\) and modifying the MSE metric. That approach is difficult to be employed in the MIMO channel because of independent power constraints. However, we provide a semi-closed form solution based on the optimization of one scalar per transmitter. Let us reformulate the \(\mathbf{P}_{\text{WMMSE}}\) problem presented in (8) as,

\[
\begin{align*}
\mathbf{E}_{ij} \left[\mathbf{d}_j - \hat{\mathbf{d}}_j \right]^2
\end{align*}
\]

\[
\begin{align*}
\mathbf{P}_{\text{WSR}} &= \min_{\mathbf{W}_j} \sum_{i,j} \text{tr}(\mathbf{W}_j \mathbf{E}_{ij}) \\
\text{s.t.} \quad &\text{tr}(\mathbf{B}_j \mathbf{B}_j^H + \mathbf{B}_j \mathbf{B}_j^H) = P_j, \quad j \in \{1, 2\}
\end{align*}
\]

where \(\mathbf{P}_j\) stands for the power constraint per transmitter and matrices \(\mathbf{W}_j\) denote the MSE-weight of the \(\mathbf{d}_j\) symbols and finally \(\mathbf{E}_{ij}\) is the expectation operator. Additionally, we define

\[
\begin{align*}
\mathbf{F}_j &= \mathbf{H}^H_j \mathbf{A}_j \mathbf{W}_j \mathbf{A}_j^H \mathbf{H}_j + \mathbf{H}^H_j \mathbf{A}_j \mathbf{W}_j \mathbf{A}_j^H \mathbf{H}_j \\
\mathbf{P}_j &= \mathbf{H}^H_j \mathbf{A}_j \mathbf{W}_j \mathbf{A}_j^H \mathbf{H}_j + \mathbf{H}^H_j \mathbf{A}_j \mathbf{W}_j \mathbf{A}_j^H \mathbf{H}_j \\
\mathbf{P}_j &= \mathbf{H}^H_j \mathbf{A}_j \mathbf{W}_j \mathbf{A}_j^H \mathbf{H}_j + \mathbf{H}^H_j \mathbf{A}_j \mathbf{W}_j \mathbf{A}_j^H \mathbf{H}_j \\
\mathbf{\Theta}_j &= \mathbf{H}^H_j \mathbf{A}_j \mathbf{W}_j \mathbf{A}_j^H \mathbf{H}_j + \mathbf{H}^H_j \mathbf{A}_j \mathbf{W}_j \mathbf{A}_j^H \mathbf{H}_j
\end{align*}
\]

where \(j \in \{1, 2\}\) and \(\mathbf{A}_j\) stands for the receive filters defined in (5) employed to decode symbols \(\mathbf{d}_j\). Assuming uncorrelated Gaussian symbols the optimization problem \(\mathbf{P}_{\text{WMMSE}}\) becomes,

\[
\begin{align*}
\min_{\mathbf{W}_j} \sum_{i,j} \text{tr}(\mathbf{W}_j \mathbf{E}_{ij}) \\
\text{s.t.} \quad &\text{tr}(\mathbf{B}_j \mathbf{B}_j^H + \mathbf{B}_j \mathbf{B}_j^H) = P_j, \quad j \in \{1, 2\}
\end{align*}
\]

Consequently, \(\mathbf{P}_{\text{WSR}}\) shown in \(\text{Error! No se encuentra el origen de la referencia.}\) can be solved through \(\mathbf{P}_{\text{WMMSE}}\) along with the proper MSE-weight matrix \(\mathbf{E}_{ij}\) for getting the MSE-weight matrix \(\mathbf{W}_j\) using (9). Then the KKT conditions for the \(\mathbf{P}_{\text{WSR}}\) are satisfied with the optimal filter for the \(\mathbf{P}_{\text{WMMSE}}\) and corresponding MMSE matrices \(\mathbf{W}_{ij}\) (see Appendix).

V. WSR MAXIMIZATION BY ALTERNATE OPTIMIZATION

Although there are different options for implementing the alternating optimization we follow the same approach employed in [10] where the MSE-weights and receivers are updated simultaneously.

**Algorithm**

1. Set \(n=0\)
2. Set \(\mathbf{B}_j^0 = \mathbf{B}_j^{n-1}\)
3. Iterate
   a. update \(n=n+1\)
   b. Compute \(\mathbf{A}_j^n | \mathbf{B}_j^{n-1}\) using (5)
   c. Compute \(\mathbf{W}_j^n | \mathbf{B}_j^{n-1}\) using (9)
   d. Compute \(\mathbf{B}_j^n | \mathbf{A}_j^n, \mathbf{W}_j^n\) using (13)
4. Until convergence

This iterative algorithm converges to a fixed point, which is proved using the same arguments as the ones given in [10], where the cost function due to the alternate minimization decreases monotonically. Hence, if we initialize it with the precoders derived from [6] to accommodate the interference alignment we can ensure a worst-case sum-rate and consequently the DoF in the high SNR region. Nevertheless, in the tests performed in the low-medium SNR (up to SNR 40 dB) the impact of the initialization is not significant.

In contrast to other type of channels, the MIMO X channel might have non-integer DoF, [4][6]. In such a case the precoding is performed through a symbol extension of 3 symbols. All the previous derivations are still valid if we work
\[
\hat{H}_j = \hat{H}_j \otimes 1_j, \\
E[\tilde{d}^T_j B^T_j \tilde{d}_j + \tilde{d}^T_j B_j \tilde{d}_j] = T \times P_j \quad j \in \{1, 2\}
\]

where $\otimes$ stands for the Kronecker operator.

VI. RESULTS

The numerical results presented in this section analyze the performance of the proposed MMSE precoder design (named X-MMSE) for the MIMO $X$ channel. Its performance will be compared with three schemes:

- **Broadcast channel with a single power constraint,** denoted in the following by DPC-BC, this scheme assumes that there is a single transmitter with $M_1+M_2$ antennas and has a maximum power constraint of $P_1+P_2$. The algorithm introduced in [15] has been used to calculate the sum-rate under the assumption of complex receivers with successive interference cancellation capability or dirty-paper coding at the transmitter.

- **MIMO $X$ channel with ZF transmitters using [6].** It assumes an equal-power allocation over the different transmitted streams and it will be denoted by X-ZF.

- **MIMO $X$ channel with ZF transmitters and additional covariance matrix for maximizing the WSR.** Notice that once the interference is aligned the MIMO $X$ channel is transformed into 4 parallel MIMO point-to-point channels. In such a case waterfilling-based precoders are employed. This scheme will be referenced by X-ZF-WF.

![Figure 2](image)

Figure 2. Average sum-rate vs SNR for the MIMO $X$ channel with $M_1=M_2=N_1=N_2=3$. 50 channel realizations. All users with the same priority.

Fig.2 and Fig.3 present the sum-rate attained by the DPC-BC, X-ZF, X-ZF-WF and X-MMSE schemes as a function of the SNR for $M_1=M_2=N_1=N_2=3$ and $M_1=5$, $M_2=8$, $N_1=6$, $N_2=7$, respectively, in a single channel realization. Both figures show the significant benefits of the X-MMSE over the X-ZF in the low-medium SNR regime in terms of SNR offset gain, while the attained DoF (slope of the sum-rate at high SNR) is the same. This latter observation is a consequence of initializing the alternate algorithm, (14), with the precoders employed by the X-ZF. In such a case the worst-sumrate case will be the one obtained by the X-ZF scheme. An additional remark is the performance of the X-ZF-WF scheme, we can see in Fig.2 and Fig.3 that there is not any significant gain by optimizing the precoders once the interference is aligned.

VII. CONCLUSIONS

Obtaining the optimal precoders that maximize the WSR of the MIMO $X$ channel with MMSE transmit and receive filters is a difficult task due to the non-convexity of the optimization problem. However, we have observed that using a simple algorithm based on the alternate optimization along with the proper initialization is enough for providing significant sum-rate gains at low-medium SNR and maintain the DoF at high SNR. Moreover, the analyzed algorithm is able to subsume the limit cases where the MIMO $X$ channel tends to MIMO-BC, dual MIMO point-to-point, MIMO-MAC and MIMO-IC.

VIII. APPENDIX

Let us formulate the dual function for the PWSR problem presented in (7),

\[
L(B_j, \lambda_j) = -\sum \mu_j R_j + \sum \lambda_j \left( \text{tr}(B^H_j B^T_j + B_j B^H_j) - P_j \right) \quad (15)
\]

with $\lambda_j$ the Lagrange multiplier associated to the $j$-th source power constraints. The transmit filter $B_{nm}$ is the solution of

\[
\frac{\partial L}{\partial B_{nm}} = -\sum \mu_j R_{nm} + \lambda_j B_{nm} = 0 \quad (16)
\]

where the different derivatives are obtained by using [14].
\[
\frac{\partial R_{nm}}{\partial B_{nn}} = \begin{cases} 
H_{nm}^H R_{w(n,m)}^{-1} H_{nm} B_{nn} E_{mn} & \text{if } (i,j) = (n,m) 
\end{cases}
\]
\[
\frac{\partial B_{nn}}{\partial B_{nn}} = \begin{cases} 
H_{nm}^H R_{w(n,m)}^{-1} H_{nm} B_{nn} E_{mn} W_{mn} E_{mn} & \text{if } (i,j) = (n,m) 
\end{cases}
\]
with \( R_{w(n,m)}^{-1} \) and \( E_q \) defined in (2) and (6), respectively. On the other hand the Lagrangian of \( P_{\text{wmmse}} \) given in (8)
\[
G(B_q, \lambda_q) = \sum_{i,j} \omega_{ij} + \omega_i + \lambda_i \left( \text{tr}(B_{ij} B_{ij}^H + B_{ij} B_{ij}^H) - P_f \right)
\]
with \( \omega_q = \text{tr}(W_q E_q) \). In order to find the transmit precoders, for example \( B_{nn} \), should satisfy,
\[
\frac{\partial G}{\partial B_{nn}} = \sum_{i,j} \frac{\partial \omega_{ij}}{\partial B_{nn}} + \lambda_{ni} B_{nn} = 0
\]
where derivatives takes into accounts [14]
\[
\frac{\partial \omega_{ij}}{\partial B_{nn}} = \begin{cases} 
H_{nm}^H R_{w(n,m)}^{-1} H_{nm} B_{nn} E_{mn} W_{mn} E_{mn} & \text{if } (i,j) = (n,m) 
\end{cases}
\]
Notice that by imposing \( W_q = \left( \mu_q / \ln 2 \right) E_q^{-1} \) then the conditions to be satisfied by \( B_{nn} \) in the problem defined in (15) are the same that the ones to be satisfied in the problem (18).

REFERENCES