CONSENSUS IN RANDOM WSNs WITH CORRELATED SYMMETRIC LINKS

Silvana Silva Pereira and Alba Pagès-Zamora

Signal Processing and Communications Group (SPCOM)
Universitat Politècnica de Catalunya - Barcelona Tech (UPC), Spain
E-mail: {silvana.silva, alba.pages}@upc.edu

ABSTRACT

In this paper we study the convergence of consensus algorithms in wireless sensor networks with random topologies and instantaneous symmetric links which exist with different probabilities independently in time but correlated in space. Aiming at minimizing the convergence time of the algorithm, we adopt an optimization criterion based on the spectral radius of a positive semidefinite matrix for which we derive a closed-form expression, and propose a general formulation for the spatial correlation among the links that allows us to compute the optimum parameters ensuring almost sure convergence. We show that our general formulation can be particularized for existing protocols found in literature, and derive additionally closed-form expressions for the optimal parameters in two particular cases of equal probability of connection for all the links.

1. INTRODUCTION

Consensus algorithms for distributed estimation of parameters in wireless sensor networks (WSNs) have received increasing attention in recent years due to a wide range of applications in several fields. In the presence of random failures caused by changes in the environment, mobility of the nodes, asynchronous sleeping periods or by randomized protocols, the topology of the network varies randomly with time and the convergence of the consensus algorithm is characterized in probabilistic terms. Important contributions on probabilistic consensus for undirected (symmetric) networks can be found in literature. For instance, Hatano and Mesbahi [1] shows convergence in probability using notions of stochastic stability for Erdős-Rényi random graphs. Assuming links with different probabilities, Kar and Moura [2] relates mean square (m.s.) convergence to the second smallest eigenvalue of the average Laplacian matrix and derives bounds on the convergence rate for random topologies. Moreover, for directed random networks Tahbaz-Salehi and Jadbabaie [3] relates the convergence to the second largest eigenvalue of the average weight matrix. Porfiri and Stilwell [4] shows that a sufficient condition for almost sure (a.s.) convergence in continuous systems is that the eigenvalues of the average Laplacian have positive real parts and that the topology varies sufficiently fast. Zhou and Wang [5] redefines the asymptotic and the per-step convergence factors from Xiao and Boyd in [6] to characterize the convergence speed in random directed networks. Fagnani and Zampieri [7] studies the asymptotic convergence rate of randomized consensus schemes based on either m.s. convergence or on the Lyapunov exponents. Based on [7] and [8], Aysal et al. [9] derives a sufficient condition for a.s. convergence of broadcast gossip algorithms, while Jakovetić et al. [10] uses the m.s. error convergence rate as an optimization criterion for assigning the weights in undirected random networks with correlated links and extends the results to the broadcast gossip algorithm.

We study the convergence of the consensus algorithm in random networks assuming instantaneous symmetric links with different probabilities which are independent in time but correlated in space. Our contribution with respect to [9,10] is that we consider the consensus model in [11] not restricted to non-negative weight matrices and provide a general formulation of the spatial correlation among links which strongly simplifies the convergence analysis. We adopt an optimization criterion based on the spectral radius of a positive semidefinite matrix for which we derive a closed-form expression, useful to establish a sufficient condition for a.s. convergence in terms of the link weights. Finally, we validate our theoretical results by deriving closed-form expressions for the optimum parameters of a known existing protocol but using our formulation.

2. PRELIMINARIES

Graph theory concepts: The information flow among the nodes of the network is described by an instantaneous undirected (symmetric) graph \( \mathcal{G}(k) = \{V, \mathcal{E}(k)\} \), where \( V \) is the constant set of vertices (nodes) and \( \mathcal{E}(k) \) is the set of edges (links) \( e_{ij} \) at time \( k \) \( \forall i, j \in \{1, \ldots, N\} \) with \( e_{ij} = e_{ji} \) [12]. \( e_{ij} \) belongs to \( \mathcal{E}(k) \) with probability \( 0 \leq p_{ij} \leq 1 \) and we assume \( p_{ii} = 0 \) \( \forall i \). The set \( \mathcal{N}_i(k) = \{j \in V : e_{ij} \in \mathcal{E}(k)\} \) is the set of neighbors of node \( i \) at time \( k \). Let \( P \in \mathbb{R}^{N \times N} \) denote the connection probability matrix with entries \( P_{ij} = p_{ij} \).
The instantaneous adjacency matrix $A(k) \in \mathbb{R}^{N \times N}$ of $G(k)$ is random and symmetric with entries: $[A(k)]_{ij} = 1$ with probability $p_{ij}$, $[A(k)]_{ij} = 0$ with probability $1 - p_{ij}$, and mean $\bar{A} = \mathbb{P}$. The degree matrix $D(k) \in \mathbb{R}^{N \times N}$ is diagonal with entries $[D(k)]_{ii} = \sum_{j=1}^{N} [A(k)]_{ij}$ and $L(k) = D(k) - A(k)$ denotes the instantaneous Laplacian. By construction, $L(k)$ is random with smallest eigenvalue $\lambda_{N}(L(k)) = 0$ with associated right and left eigenvector $\mathbf{I} \in \mathbb{R}^{N \times 1}$, an all-ones vector of length $N$. If $G(k)$ is connected, $\lambda_{N}(L(k))$ has algebraic multiplicity one and $L(k)$ is an irreducible matrix. In the following, the average graph over time is assumed connected with associated Laplacian $\bar{L} = \bar{D} - \bar{P}$.

Spatially correlated links: The entries of $A(k)$ are assumed independent over time but correlated in space, and this information is provided by the matrix $C \in \mathbb{R}^{N \times N}$ with entries

$$C_{st} = \begin{cases} 0, & s = t \\ \mathbb{E}[a_{ij}a_{qr}] - p_{ij}p_{qr}, & s \neq t \end{cases}$$

for distinct nodes $i, j, q, r$ and $a_{ij} = [A(k)]_{ij}$, (the time indexing is omitted since it does not affect the computations). That is, the off-diagonal entries of $C$ are the covariance between links $e_{ij}$ and $e_{qr}$, whereas the diagonal entries are set equal to 0. This matrix will be used to derive closed-form expressions in section 4.

3. CONSENSUS IN RANDOM NETWORKS

Consider a WSN composed of $N$ nodes indexed with $i \in \{1, \cdots, N\}$ and let $x(k) \in \mathbb{R}^{N \times 1}$ denote the vector of all states at time $k$, initialized at time $k = 0$ with the value of the measurements. The evolution of $x(k)$ can be written in matrix form as follows

$$x(k + 1) = W(k)x(k), \quad \forall k > 0$$

where $W(k) \in \mathbb{R}^{N \times N}$ is the instantaneous weight matrix at time $k$, with a nonzero entry $ij$ whenever $e_{ij} \in \mathcal{E}(k)$, $\forall i, j \in \{1, \cdots, N\}$. The matrices in the set $\mathcal{S} = \{W(k), \forall k \geq 0\}$ are symmetric i.i.d. and have at least one eigenvalue equal to one with associated left and right eigenvector $\mathbf{1}$ such that

$$W(k)\mathbf{1} = \mathbf{1}, \quad \mathbf{1}^T W(k) = \mathbf{1}^T, \quad \forall k. \quad (3)$$

A right eigenvector $\mathbf{1}$ implies that after reaching a consensus, the network will remain in consensus, and a left eigenvector $\mathbf{1}$ implies that the average is preserved from iteration to iteration. The matrices in $\mathcal{S}$ are irreducible in expectation, i.e. $\lambda_{1}(\mathbf{W}) = 1$ with algebraic multiplicity one. This is important to guarantee that the network reaches a probabilistic consensus asymptotically. The average weight matrix satisfies also $\bar{W}\mathbf{1} = \mathbf{1}$ and $\mathbf{1}^T \bar{W} = \mathbf{1}^T$. In order to find the rate at which the network reaches a consensus, in the following section we present a criterion for reducing the convergence time which establishes a sufficient condition for a.s. convergence under these topology conditions.

4. A SUFFICIENT CONDITION FOR A.S. CONVERGENCE

Following the line of previous contributions [7–10], consider the distance of $x(k)$ to the average consensus vector $x_{a} = 11^T x(0)/N$, defined as

$$d(k) = x(k) - x_{a} = (I - J)x(k)$$

where $I \in \mathbb{R}^{N \times N}$ is the identity matrix and $J = \frac{1}{N} 11^T$. Using the properties of $W(k)$ we have


Further, the expected norm of $d(k + 1)$ given $d(k)$ is

$$\mathbb{E} \left[ |d(k + 1)|^2 \right] = \mathbb{E} (d(k)^T \mathbb{W} d(k) \leqslant \lambda_{1}(W) |d(k)|^2$$

where

$$\mathbb{W} = \mathbb{E} \left[ W(k)^T (I - J)W(k) \right] = \mathbb{E} \left[ W(k)^T W(k) \right] - J$$

and we have considered that $(I - J)$ is symmetric and idempotent. Repeatedly conditioning and replacing iteratively for $d(k)$ we obtain

$$\mathbb{E} \left[ |d(k)|^2 \right] \leqslant \lambda_{1}^2(W) |d(0)|^2 \quad (5)$$

$\lambda_{1}(W)$ in the expression above can be identified as the asymptotic m.s. convergence factor defined in [5], which sets necessary and sufficient conditions for m.s. asymptotic stability but which are only sufficient for a.s. stability. The right hand side in (5) is an upper bound for $\mathbb{E} \left[ |d(k)|^2 \right]$, and the minimization of the convergence rate for this upper bound is the optimization criterion chosen to reduce the convergence time of (2). The following result is shown in [5,9,10]:

Lemma 1. The consensus algorithm in (2) with weight matrices satisfying (3) converges almost surely to a consensus value if

$$\lambda_{1}(W) < 1$$

where $W$ is the matrix defined in (4) and $\lambda_{1}(\cdot)$ denotes its largest eigenvalue.

A necessary and sufficient condition for the convergence of $d(k)$ to a zero vector can be derived from [8] and reads

$$\lambda_{1}(\mathbb{E}[W(k) \otimes W(k)] - \tilde{J}) < 1 \quad (7)$$

where $\otimes$ denotes Kronecker product and $\tilde{J} = 11^T / N^2$ with $1 \in \mathbb{R}^{N \times 1}$. However, the criterion of minimizing $\lambda_{1}(W)$ in (6) is easier to evaluate than (7), and is a convex optimization problem, as shown in [10] (Lemma 3). The next step consists in analyzing the matrix $\mathbb{W}$ focusing on weight matrices satisfying (3) and of the form

$$W(k) = I - \epsilon L(k), \quad \forall k \geq 0, \quad (8)$$


where $L(k)$ is the Laplacian at time $k$ and $\epsilon$ is chosen to satisfy convergence conditions. Since we focus on reducing the convergence time of the algorithm, we aim at finding the time-invariant value of $\epsilon$ that minimizes $\lambda_1(W)$. The main contribution of this paper is presented in the following theorem:

**Theorem 1.** Consider the consensus algorithm in (2) with $N$ nodes, $W(k)$ defined in (8) and satisfying (3), and spatially correlated random links. The matrix $W$ in (4) has a closed-form expression given by

$$W = \left( \bar{L}^2 + 2(\bar{L} - \bar{L}) + R \right) \epsilon^2 - 2\bar{L}\epsilon + I - J$$  \hspace{1cm} (9)

where $\bar{L}$ is the Laplacian of the expected underlying graph,

$$\bar{L} = \bar{D} - \bar{P} \odot \bar{P},$$

$\bar{D}$ is diagonal with entries $\bar{D}_{ii} = [(\bar{P} \odot \bar{P})_{i}]$, and connection probability matrix $\bar{P}$, $\odot$ denotes the Schur product, and $R$ is an $N \times N$ symmetric matrix built with the covariance terms in $C$ as follows

$$R_{mm} = a_m^T C a_m, \quad R_{mn} = \sum_i e_{in}^T C e_{im} - a_i^T C e_{mn} - a_i^T C e_{mm},$$  \hspace{1cm} (10)

with

$$[a_m]_i = \begin{cases} 1 & i = N(m-1)+j \quad j = 1, \cdot \cdot \cdot , N \\ 0 & \text{otherwise} \end{cases}$$

$$[b_m]_i = \begin{cases} 1 & i = N(j-1)+m \quad j = 1, \cdot \cdot \cdot , N \\ 0 & \text{otherwise} \end{cases}$$

$$[e_{mn}]_i = \begin{cases} 1 & i = N(m-1)+n \\ 0 & \text{otherwise} \end{cases}$$

**Proof.** The proof is omitted due to lack of space and included in the extended version [13].

Note that $\bar{L}$ in Theorem 1 is a Laplacian matrix whose off-diagonal entries are the squared entries of $P$. However, $\bar{L}$ is not diagonalized by the same set of eigenvectors that diagonalize $L$ and therefore a closed-form expression for $\lambda_1(W)$ can not be derived, except for the special case of links with equal probability of connection, as we will see in Section 5. It is not difficult to show that the set of values of $\epsilon$ for which $\lambda_1(W)$ satisfies (6) and the value of $\epsilon$ for which $\lambda_1(W)$ is minimum exist, and both are positive [13]. Moreover, the $\epsilon$ minimizing $\lambda_1(W)$ can be computed using the subgradient algorithm as suggested by [8, 10]. For the simulations in this paper we have used an exhaustive search over a closed interval. In the following section, we show that our general formulation in (9) for undirected random networks can be particularized to obtain the closed-form expressions in [10], and derive in addition closed-form expressions for topologies with instantaneous links of equal probability.

### 5. PARTICULARIZING THE MAIN RESULTS

**Correlated links in Erdős-Rényi topology:** We study the behavior of $\lambda_1(W)$ when the expected underlying graph has an Erdős-Rényi random topology with probability of connection $p$. We consider the case of equal covariance among links $s$ and $t$ as derived in (1) for distinct nodes $i, j, q, r \in \{1, \cdot \cdot \cdot , N\}$, where the matrix $C$ has entries

$$C_{s,t} = \begin{cases} 0, & s = t \\ p(1-p)v, & s \neq t, \quad 0 < v < 1 \end{cases}$$  \hspace{1cm} (11)

The following Lemma gives closed-form expressions for the $\epsilon^*$ that minimizes $\lambda_1(W)$ and its dynamical range for these topologies.

**Lemma 2.** Consider the consensus algorithm in (2) with $N$ nodes, $W(k)$ defined in (8) and satisfying (3), $P_{ij} = p \forall i \neq j$ and spatially correlated random links with $C_{s,t} = p(1-p)v$; $0 < v < 1$ for $s \neq t$, and $s, t$ defined in (1). The value of $\epsilon$ minimizing $\lambda_1(W)$ is given by

$$\epsilon^* = \frac{1}{Np + (1-p)(2 + v(N-2))}$$  \hspace{1cm} (12)

and the dynamical range for $\epsilon$ is

$$\epsilon \in \left( 0, \frac{2}{Np + (1-p)(2 + v(N-2))} \right).$$  \hspace{1cm} (13)

**Proof.** When the off-diagonal entries of the matrix $P$ are all equal and nonzero, the expected Laplacian is $\bar{L} = p(1-J)$ and $\bar{L} = p^2(1-J)$. $\bar{L}$ has one eigenvalue 0 with algebraic multiplicity one, and $Np$ with algebraic multiplicity $N-1$, whereas $\bar{L}$ has 0 with algebraic multiplicity one and $Np^2$ with algebraic multiplicity $N-1$. The entries of $R$ and its eigenvalues can be easily computed, since the nonzero entries of $C$ in (11) are all equal. According to (10) we have

$$R_{mm} = (N-1)(N-2)p(1-p)v$$

$$R_{mn} = -(N-2)p(1-p)v$$

such that

$$R = N(N-2)p(1-p)v(I-J).$$

The eigenvalues of $R$ are therefore 0 with algebraic multiplicity one and $N(N-2)p(1-p)v$ with algebraic multiplicity $N-1$. Remark that $\bar{L}, R$ and $\bar{L}$ have the same structure and are diagonalized by the same set of eigenvectors. The matrix $W$ in (9) has two distinct eigenvalues: 0 and

$$\lambda_i(W) = (Np + (1-p)(2 + v(N-2))) Np p^2 - 2Np p + 1$$  \hspace{1cm} (14)

for $i = 1, \cdot \cdot \cdot , N-1$. Therefore, the minimum of $\lambda_1(W)$ is attained when the function above reaches its minimum. Taking the derivative of (14) and solving for $\epsilon$ we obtain (12). Finally, since the function in (14) is quadratic on $\epsilon$, the derivation of (13) is straightforward, which completes the proof. ∎
The expression for the link weight in (12) coincides with the expression derived by Jakovetić et al. in [10] for these topologies and considering the correlation model in (11), showing that the closed-form results in [10] can be particularized using our expression in (9).

Uncorrelated links with equal probability: We study the behavior of $\lambda_1(\mathcal{W})$ when the expected graph has either a random geometric topology or an Erdős–Rényi topology, both with equal probability of connection $p$ for all the links. Note that when the links are spatially uncorrelated, $\mathbf{R}$ vanishes from the quadratic term in (9). The following Lemma refers to the case of a WSN with random geometric expected topology.

**Lemma 3.** Consider the consensus algorithm in (2) in a WSN with $N$ nodes and random geometric expected topology, $\mathbf{W}(k)$ defined in (8) and satisfying (3), and spatially uncorrelated random links with equal probability of connection $p$. The value of $\epsilon$ minimizing $\lambda_1(\mathcal{W})$ is given by

$$\epsilon^* = \min\left\{ \frac{2}{\lambda_{N-1}(\mathbf{L})} \right\}$$

and its dynamical range is given by

$$\epsilon \in \left( 0, \frac{2}{\lambda_{N-1}(\mathbf{L}) + 2(1-p)} \right)$$

**Proof.** We provide here a reduced version of the proof in [13]. Let’s express the eigenvalues of $\mathcal{W}$ as quadratic functions

$$f_i(\epsilon) = \left( \lambda_i(\mathbf{L}) + 2(1-p) \right) \lambda_i(\mathbf{L}) \epsilon^2 - 2\lambda_i(\mathbf{L}) \epsilon + 1, \quad (17)$$

for $i = \{1, \ldots, N-1\}$, where the subindices are in one to one correspondence with the ordering of the eigenvalues of $\mathbf{L}$, but not with the eigenvalues of $\mathcal{W}$. As the magnitude of $\lambda_i(\mathbf{L})$ increases, the terms in $\epsilon$ in (17) increase and the quadratic curves become narrower with a vertex approaching the abscissa. Note that at $\epsilon = 0$, $f_i(\epsilon) = 1 \forall i$, and the slope at that point is $-2\lambda_i(\mathbf{L})$. The curve with slowest decay corresponds to $i = N-1$, such that in the proximity of $\epsilon = 0$, $\lambda_1(\mathcal{W})$ is given by $f_{N-1}(\epsilon)$. We evaluate next the intersections of $f_{N-1}(\epsilon)$ with the remaining curves, which occur at the point $2/\left( \lambda_{N-1}(\mathbf{L}) + \lambda_j(\mathbf{L}) + 2(1-p) \right)$ for $j \in \{1, \ldots, N-2\}$. The first intersection occurs at $j = 1$, giving $\epsilon_{\text{int}} = 2/\left( \lambda_{N-1}(\mathbf{L}) + \lambda_1(\mathbf{L}) + 2(1-p) \right)$. Observe that no other curve intersects $f_1(\epsilon)$ for $\epsilon < \epsilon_{\text{int}}$ since $2/\left( \lambda_1(\mathbf{L}) + \lambda_j(\mathbf{L}) + 2(1-p) \right) < \epsilon_{\text{int}}$. Summing up, $\lambda_1(\mathcal{W})$ is given by $f_{N-1}(\epsilon)$ for $\epsilon \in (0, \epsilon_{\text{int}})$, whereas from $\epsilon_{\text{int}}$ it is given by $f_1(\epsilon)$. Finally, we must check if the minimum of $f_{N-1}(\epsilon)$, which occurs at $\epsilon_{\text{min}} = 1/\left( \lambda_{N-1}(\mathbf{L}) + 2(1-p) \right)$, is attained before or after the intersection with $f_1(\epsilon)$. Therefore, the optimum $\epsilon^*$ will be given by $\min\{\epsilon_{\text{int}}, \epsilon_{\text{min}}\}$. In order to specify the dynamical range of $\epsilon$, observe that the curve for $f_1(\epsilon)$ is the narrower one, and since no other curves are crossing $f_1(\epsilon)$ for $\epsilon > \epsilon_{\text{int}}$, the upper bound for $\epsilon$ in (16) is found simply equating $f_1(\epsilon)$ to one and solving for $\epsilon$. \qed

Note that when the probability of connection $p = 1$, we obtain a deterministic system and the closed-form expressions in (15) and (16) coincide with the expressions for fixed topologies derived in [6]. Lemma 4 below refers to the case of an Erdős–Rényi expected graph with probability $p$.

**Lemma 4.** Consider the consensus algorithm in (2) with $N$ nodes, $\mathbf{W}(k)$ defined in (8) and satisfying (3), and spatially uncorrelated random links with $P_{ij} = p$ for all $i \neq j$. The value of $\epsilon$ minimizing $\lambda_1(\mathcal{W})$ is given by

$$\epsilon^* = \frac{1}{Np + 2(1-p)}$$

and its dynamical range is given by

$$\epsilon \in \left( 0, \frac{2}{Np + 2(1-p)} \right).$$

**Proof.** We evaluate next the intersections of $f_{N-1}(\epsilon)$ with the remaining curves, which occur at the point $2/\left( \lambda_{N-1}(\mathbf{L}) + \lambda_j(\mathbf{L}) + 2(1-p) \right)$ for $j \in \{1, \ldots, N-2\}$. The first intersection occurs at $j = 1$, giving $\epsilon_{\text{int}} = 2/\left( \lambda_{N-1}(\mathbf{L}) + \lambda_1(\mathbf{L}) + 2(1-p) \right)$. Observe that no other curve intersects $f_1(\epsilon)$ for $\epsilon > \epsilon_{\text{int}}$ since $2/\left( \lambda_1(\mathbf{L}) + \lambda_j(\mathbf{L}) + 2(1-p) \right) < \epsilon_{\text{int}}$. Summing up, $\lambda_1(\mathcal{W})$ is given by $f_{N-1}(\epsilon)$ for $\epsilon \in (0, \epsilon_{\text{int}})$, whereas from $\epsilon_{\text{int}}$ it is given by $f_1(\epsilon)$. Finally, we must check if the minimum of $f_{N-1}(\epsilon)$, which occurs at $\epsilon_{\text{min}} = 1/\left( \lambda_{N-1}(\mathbf{L}) + 2(1-p) \right)$, is attained before or after the intersection with $f_1(\epsilon)$. Therefore, the optimum $\epsilon^*$ will be given by $\min\{\epsilon_{\text{int}}, \epsilon_{\text{min}}\}$. In order to specify the dynamical range of $\epsilon$, observe that the curve for $f_1(\epsilon)$ is the narrower one, and since no other curves are crossing $f_1(\epsilon)$ for $\epsilon > \epsilon_{\text{int}}$, the upper bound for $\epsilon$ in (16) is found simply equating $f_1(\epsilon)$ to one and solving for $\epsilon$. \qed

6. SIMULATION RESULTS

The analytical results derived in this paper are supported here with computer simulations over a network with random geometric expected topology. We consider $N = 20$ nodes deployed uniformly at random in a unit square and fixed position, where two nodes can communicate only if the euclidean distance between them is smaller than $0.37$. For the verification of the closed-form expressions in (9), we choose a very general model with different probabilities of connection and different correlation among pairs of links. Therefore, the instantaneous links among neighboring nodes $i$ and $j$ are generated as correlated Bernoulli random variables (r.v.’s) with different probabilities of connection $p_{ij}$, chosen uniformly at random between $[0,1]$. For the spatial correlation we consider the autoregressive model in [14], where the entries of the covariance matrix are given by

$$C_{st} = \left\{ \begin{array}{ll} 0, & s = t \\ \psi(1-\psi)^{t-s} & s \neq t, \end{array} \right.$$ 

and $\zeta_s = p_{ij}/\psi$, $\zeta_t = p_{qr}/\psi$ with $s$ and $t$ as defined in (1), $\psi = 0.3$ and $\psi = \max\{P\} = 0.98$. The entries of $\mathbf{X}(0)$ are modeled using Gaussian r.v.’s with mean $x_m = 20$ and variance $\sigma^2 = 5$. A total of $10,000$ independent realizations were run to obtain $E \left[ \mathbf{d}(k) \right] \hat{[3]}$, where $\mathbf{P}$ was kept fixed while a new $\mathbf{L}(k)$ was generated at each iteration. The expression in (9) has been verified with simulations and the difference measured between the theoretical and the empirical $\lambda_1(\mathcal{W})$ lies around $5 \times 10^{-3}$ (see Fig. 1).
Three different values of $\epsilon$ were chosen to verify the results. Fig. 2 shows $E[\|d(k)\|^2]$ in logarithmic scale as a function of the iteration index $k$ for: $\epsilon^* = 0.2412$ (solid line), $\epsilon = 1/(N-1) = 0.0526$ (dashed line) and $\epsilon = 0.2$ (dashed-dotted line). We conclude that the minimization of $\lambda_1(W)$ is a good design criterion to reduce the convergence time of the consensus algorithm in random WSNs with correlated symmetric links. Moreover, we have observed that for values of $\epsilon$ that result in $\lambda_1(W) > 1$, the algorithm does not converge, verifying Lemma 1.

7. CONCLUSIONS

We have studied the convergence of the consensus algorithm in random topologies with spatially correlated symmetric links, where a useful criterion for the minimization of the convergence time has been studied. This criterion is based on the spectral radius of a positive semidefinite matrix for which we have derived a closed-form expression, and states a sufficient condition for almost sure convergence. A general formulation for the spatial correlation among the links has been proposed, and closed-form expressions have been derived which greatly simplify the derivation of this convergence condition in terms of the link weights. The results have been validated particularizing our general expressions for a known existing protocol, and we have in addition derived closed-form expressions for the dynamical range and the optimum link weight for topologies with links of equal probability of connection. The performance improvement has been verified through computer simulations of a general case with different probabilities of connection for all the links and different correlations among pairs of possible existing links.

8. REFERENCES


Fig. 1. Theoretical and empirical $\lambda_1(W)$ as a function of $\epsilon$.

Fig. 2. Empirical $E[\|d(k)\|^2]$ in logarithmic scale as a function of $k$ for three different values of $\epsilon$ in a network with correlated symmetric random links.